$\gamma_0 = 1.577$. Figure 5 shows the law of distribution of $\gamma(t)$ as a function of t when m = 1.5 and 2 (curves 1 and 2, respectively). It should be noted that this function is almost linear. In addition, the set under the die will be larger for lower nonlinearity indexes m if other conditions are equal.

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NUMERICAL ASYMPTOTIC SOLUTION OF STRENGTH AND VIBRATIONS PROBLEMS OF THIN SHELLS OF REVOLUTION

S. V. Stepanenko

UDC 539.3

For thin shells of revolution whose middle surface has a nonnegative Gaussian curvature, a numerical analytical approximate solution is constructed for the class of linear boundary-value problems allowing of separation of variables.

It is known that the solution of each such problem decomposes into a slowly varying part and a solution of edge effect type. On this basis, a method of construction the approximate solution of the problem is proposed in [1, 2], where it is proposed to seek the slowly varying part of the solution by a numerical method, and the edge effects by an asymptotic method. On the basis of this method, an algorithm is constructed in this paper, which can be applied to a broader class of problems as compared to [1] because of utilization of the method of elimination in the boundary conditions [3]. As an illustration of the method, solutions are presented for a number of strength and vibrations problems for shells of different geometries.

1. Many strength and vibration problems for elastic shells of revolution reduce to seeking solutions of a particular kind

$$u_\sigma^{mn} = \exp(i\omega_m t + inx_2)\,U_\sigma^{mn}(x_1),$$
$$w^{mn} = \exp(i\omega_m t + inx_2)W^{mn}(x_1). \quad (1.1)$$

Here t is the time; x_1, x_2, orthogonal coordinates of the shell middle surface; $i = \sqrt{-1}$; ω_m, real integers; and m and n, integers, the subscript σ takes on the values 1 and 2; u_1, u_2, w, displacements in the x_1, x_2 directions and along the external normal. The well-developed apparatus of shallow shell theory [4] can be applied to describe solutions of the form (1.1) with $n \geqslant 4$. In the absence of tangential components of the surface forces, a force function $\Psi(x_1, x_2, t)$ is introduced. By virtue of (1.1) we will have

$$\varphi^{mn}(x_1, x_2, t) = \exp(i\omega_m t + inx_2)\Phi^{mn}(x_1). \quad (1.2)$$

The system of governing equations of shallow shell theory after making the system operators dimensionless and substituting functions from (1.1) and (1.2) becomes

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 130-134, March-April, 1983. Original article submitted January 19, 1982.

0021-8944/83/2402-0256$07.50

$$\varepsilon^2\Delta\Delta\Phi_*^{mn}(x) - \Delta_k W_*^{mn}(x) = 0,$$
$$\varepsilon^2\Delta\Delta W_*^{mn}(x) + \Delta_k\Phi_*^{mn}(x) = Q_*^{mn}(x), \quad (1.3)$$

where

$$\Delta = \frac{1}{AB_*}\left[\frac{d}{dx}\left(\frac{B_*}{A}\frac{d}{dx}\right) - n^2\frac{A}{B_*}\right];\ \Delta_k = \frac{1}{AB_*}\left[\frac{d}{dx}\left(\frac{B_*K_2^*}{A}\frac{d}{dx}\right) - n^2\frac{AK_1^*}{B_*}\right];$$
$$x = x_1/l;\ B_* = B/l;\ K_1^* = K_1 l;\ K_2^* = K_2 l;\ W_*^{mn} = h^{-1}W^{mn};$$
$$\Phi_*^{mn} = [12(1-\nu^2)]^{\frac{1}{2}}(Eh^3)^{-1}\Phi^{mn};\ Q_*^{mn} = l^3[12(1-\nu^2)]^{\frac{1}{2}}(Eh^3)^{-1}Q^{mn};$$

$\varepsilon^2 = (h/l)[12(1-\nu^2)]^{-1/2}$ is a small parameter; x, longitudinal coordinate $x \in [0, 1]$; l, shell length; h = const, thickness; A(x), B(x) and $K_1(x)$, $K_2(x)$, Lamé parameters and the principal curvatures of the middle surface; ν, Poisson ratio; and E, elastic modulus. The form of the function Q^{mn} depends on the formulation of the problem. For a strength problem $Q^{mn} = Z^{mn}$, where $z^{mn} = \exp(i\omega_m t + inx_2)Z^{mn}(x_1)$ is the normal surface load, while for a natural vibrations problem $Q^{mn} = \rho h\omega_m^2 W^{mn}$, where ρ is the density of the material.

In a correct formulation, the boundary-value problem for system (1.3) should have eight boundary conditions, four on each end-face. In this paper, w^{mn}; ϑ_1^{mn} or M_1^{mn}; u_1^{mn} or N_1^{mn}; u_2^{mn} or S^{mn} given on the shell end-faces, which corresponds to four framing versions or four hinge versions, are considered as boundary conditions. Here M_1 is the longitudinal bending moment, ϑ_1 is the angle of rotation in the meridian plane, and N_1 and S are the normal and shear forces. Without limiting the generality, only homogeneous boundary conditions will be considered.

Let us write system (1.3) in operator form, whereupon we introduce the vector of the solution $\mathbf{Y}(x) = (\Phi_*^{mn}(x), W_*^{mn}(x))$ and the vector of the right side $\mathbf{P}(x) = (0, Q_*^{mn}(x))$:

$$\varepsilon^2 L(x)\mathbf{Y}(x) - M(x)\mathbf{Y}(x) + N(x)\mathbf{Y}(x) = \mathbf{P}(x), \quad (1.4)$$

where L(x), M(x), N(x) are 2 × 2 matrices with the elements

$$L_{11} = L_{22} = \frac{1}{AB_*}\frac{d}{dx}\left\{\frac{B_*}{A}\frac{d}{dx}\left[\frac{1}{AB_*}\frac{d}{dx}\left(\frac{B_*}{A}\frac{d}{dx}\right)\right]\right\};\ L_{12} = L_{21} = 0;$$
$$M_{11} = M_{22} = \frac{\varepsilon^2 n^2}{AB_*}\left\{\frac{d}{dx}\left[\frac{B_*}{A}\frac{d}{dx}\left(\frac{1}{B_*^2}\right)\right] + \frac{1}{B_*^2}\frac{d}{dx}\left(\frac{B_*}{A}\frac{d}{dx}\right)\right\};$$
$$-M_{12} = M_{21} = \frac{1}{AB_*}\frac{d}{dx}\left(\frac{B_*K_2^*}{A}\frac{d}{dx}\right);\ N_{11} = N_{22} = \frac{\varepsilon^2 n^4}{B_*^4};$$
$$-N_{21} = N_{12} = \frac{n^2K_1^*}{B_*^2}.$$

After replacement of the boundary quantities by means of the desired functions, the boundary conditions can be written in the following form with the vector representation of the solution taken into account

$$D_{1j}(x)\mathbf{Y}(x)|_{x=x_j} = 0,\ D_{2j}(x)\mathbf{Y}(x)|_{x=x_j} = 0\ (j = 0,1), \quad (1.5)$$

where D_{1j} and D_{2j} are 2 × 2 matrices with differential operators as elements in the general case. It is considered that the matrix D_{2j} contains higher-order operators (up to the third, inclusive, in certain versions of the boundary conditions). Here and henceforth, if not specially stipulated otherwise, $x_0 = 0$, $x_1 = 1$.

2. The theory of representation of the solution in the form of a series in a small parameter developed in [5, 6] for ordinary differential equations can be applied formally to solve the problem (1.4), (1.5). The solution of problem (1.4), (1.5) is sought in the form

$$\mathbf{Y} = \mathbf{Y}_0 + \varepsilon\mathbf{Y}_1 + \ldots + \varepsilon^\rho(\mathbf{Y}_{00} + \varepsilon\mathbf{Y}_{01} + \ldots) + \varepsilon^\sigma(\mathbf{Y}_{10} + \varepsilon\mathbf{Y}_{11} + \ldots). \quad (2.1)$$

The first group of terms represents an asymptotic series of the slowly varying part of the solution, while the second and third groups are asymptotic series for the edge effects. The quantities ρ and σ (on the order of the edge effects) are determined by the kind of boundary conditions. For small ε it is sufficient to keep only the principal terms of the series for the complete solution; hence, it is possible to stop after the first step of the iteration process proposed in [5]. The first term in the series of the slowly varying part of the solution is sought as the solution of the equation

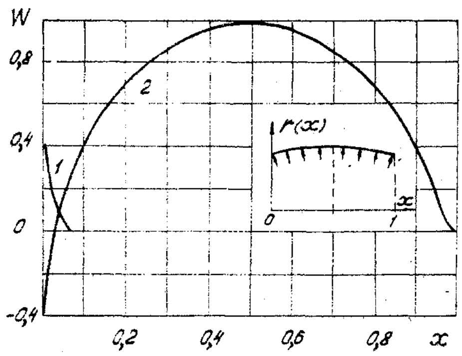

Fig. 1

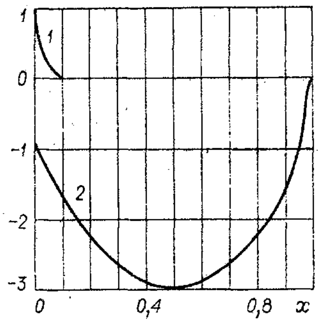

Fig. 2

$$-M(x)\mathbf{Y}_0(x) + N(x)\mathbf{Y}_0(x) = \mathbf{P}(x). \tag{2.2}$$

The principal terms of the edge effects are determined from the equations

$$\varepsilon^2 L(x_i)\mathbf{Y}_{i0}(x) - M(x_i)\mathbf{Y}_{i0}(x) = 0 \ (i = 0, 1) \tag{2.3}$$

with the additional condition

$$\mathbf{Y}_{00}(x) \underset{x\to\infty}{\longrightarrow} 0, \ \mathbf{Y}_{10}(x) \underset{x\to-\infty}{\longrightarrow} 0. \tag{2.4}$$

The process of freezing the coefficients of (2.3) at the boundary points is explained by the fact that the solutions (2.3) satisfying the conditions (2.4) differ substantially from zero only in the neighborhood of a boundary point commensurate with the magnitude of the small parameter ε. For shells with weak variability of the generatrix near the end-faces, the values of the derivatives of the coefficients multiplied by ε can be neglected as compared with the value of the coefficients themselves. The solutions of (2.3) satisfying conditions (2.4) are written in the form

$$\mathbf{Y}_{i0} = \exp\left[(-1)^{1-i}\alpha_i(x-x_i)\right]\begin{pmatrix}\cos\beta_i(x-x_i) & (-1)^{1-i}\sin\beta_i(x-x_i)\\ (-1)^i\sin\beta_i(x-x_i) & \cos\beta_i(x-x_i)\end{pmatrix}\begin{pmatrix}C_{i1}\\ C_{i2}\end{pmatrix} (i = 0, 1), \tag{2.5}$$

$$\alpha_i = \left[(a_i^2 + b_i^2)^{1/2} + b_i\right]^{1/2}, \ \beta_i = \left[(a_i^2 + b_i^2)^{1/2} - b_i\right]^{1/2},$$

$$a_i = \frac{A^2(x_i)K_2^*(x_i)}{2\varepsilon^2}, \ b_i = \frac{n^2A^2(x_i)}{B_*^2(x_i)}, \ C_{ij} = \text{const}\,(j = 1, 2).$$

To formulate the boundary conditions on the solution of (2.2), we should proceed as follows: substitute the representation of the solution in the form (2.1) with a specific kind of $\mathbf{Y}_{i0}$ from (2.5) into (1.5), for large α_i solutions of boundary-layer type can be considered zero at opposite boundary points. We solve the system obtained for C_{ij} and obtain two groups of equalities

$$\begin{pmatrix}C_{i1}\\ C_{i2}\end{pmatrix} = D_{3i}(x)\,\mathbf{Y}_0(x)|_{x=x_i} \quad (i = 0, 1); \tag{2.6}$$

$$D_{4i}(x)\,\mathbf{Y}_0(x)|_{x=x_i} = 0 \quad (i = 0, 1). \tag{2.7}$$

Here D_{3i}, D_{4i} are 2×2 matrices with differential operators as elements (in the general case). The specific form of the matrices D_{3i} and D_{4i} is determined by the form of the matrices D_{1i} and D_{2i}. In practical computations it is more expedient to construct the matrix D_{3i} from the matrix D_{1i}. The equalities (2.7) have the meaning of boundary conditions for (2.2), while the constants of the solutions (2.5) are determined from (2.6) after having solved the problem (2.2) and (2.7).

3. For the numerical integration of the limit problem (2.2) and (2.7), whose order is two below the original, any standard method [7] can be utilized. However, as is shown in [8], among the solutions of the problem (2.2) and (2.7) exponentially damped components are kept, which have a quite definite edge effect nature in problems with large n or lK_2^*. In this connection, the Godunov [9] method of discrete orthogonalization, which is highly recommended and extensively utilized in shell theory problems, is used in this paper. After numerical integration of the limit problem (2.2) and (2.7), the constants of the solutions

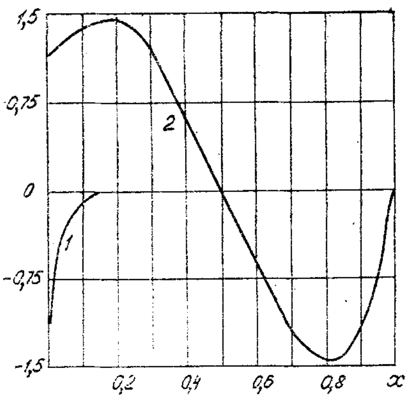

Fig. 3

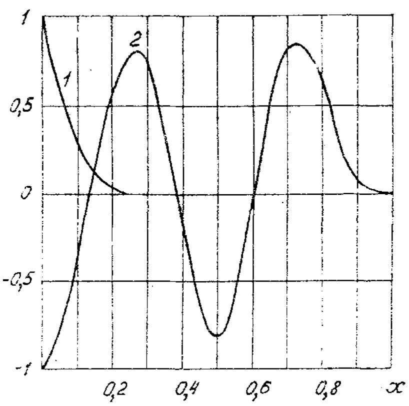

Fig. 4

(2.5) are determined from the equalities (2.6). The total solution is an approximation of the solution of the initial problem and according to estimates [5], the error in the solution constructed is of the order of $O(\varepsilon)$. In order to improve the approximation, several steps of the iteration process should still be executed [5], which actually reduce to solving the same problems for (2.2) and (2.3) with right sides that are functions of the solutions already constructed. Since the parameter ε is of the order of $O(10^{-2})$ in real thin-walled shells, and performing the next iterations causes no substantial difficulties in comparison with those elucidated above, the procedure to construct the next approximations will not be considered.

Considered as an application is a shell with the generator $r(x) = 0.45 + 0.2x - 0.2x^2$, $l/h = 200$ with the boundary conditions $w = 0$, $\vartheta_1 = 0$, $N_1 = 0$, $S = 0$ on both end-faces. The material characteristics are $\nu = 0.3$, $E = 2 \cdot 10^{10} \cdot 9.81$ N/m^2. Displayed in Fig. 1 is the normal deflection that occurs in a shell subjected to normal internal pressure with the oscillation index $n = 15$. The deflection is symmetric relative to the middle of the segment [0, 1], hence components of the solution are displayed in the left side of Fig. 1 (1 is the edge effect, 2 is the result of the numerical solution), and the complete solution is on the right. The natural vibration modes of this same shell with the same boundary conditions for $n = 4$ are displayed in Figs. 2-4 for values of the frequency parameters $\lambda = \rho l^2 \omega_m^2 / E\varepsilon^2$ equal to 221.22, 510.05, 1007.04, respectively. The modes are decomposed into components in the left sides of Figs. 2-4, and total solutions are presented in the right. The edge effect and the numerical result are denoted by the numbers 1 and 2.

An advantage of this method of approximately solving the problem is the fact that the problem is solved numerically whose order of the differential operator is two below the original, while there are no edge effects among the solutions of this problem which are generated by the thin-walledness of the structure. The method will be all the more effective, the higher the thin-walledness of the shell under consideration.

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STABILITY OF THIN SHALLOW SHELLS OF NEGATIVE GAUSSIAN CURVATURE

V. M. Ermolenko and V. M. Kornev UDC 539.31

The density of eigennumbers in stability problems of shells with positive Gaussian curvature is examined in [1-3]. An interpretation of the results obtained was proposed that permits relating the density of the initial section of the spectrum to the shell responsiveness to small perturbations during experiment, and also to imperfections in the geometric shape of the shell. Investigation of the spectrum in problems studied less, stability problems of shells of negative Gaussian curvature, is natural. Of greatest interest are shells of negative Gaussian curvature that are almost cylindrical.

The system of stability equations of shallow shells whose radii are almost constant has the form [4]

$$(\mathrm{Eh})^{-1}\,\nabla^2\nabla^2\varphi-\Delta_k^2 w=0,\ D\nabla^2\nabla^2 w+\Delta_k^2\varphi=\sigma\nabla^2\nabla^2(\alpha_1 w_{,xx}+\alpha_2 w_{,yy}),$$
$$\sigma\alpha_1=-T_1,\ \sigma\alpha_2=-T_2,\ \nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},\ \Delta_k^2=\frac{1}{R_2}\frac{\partial^2}{\partial x^2}+\frac{1}{R_1}\frac{\partial^2}{\partial^2 y^2},$$

where x, y are Cartesian coordinates; w(x, y), normal deflection; φ(x, y), stress function; T_1, T_2, forces in the shell middle surface; and $R_1\approx$ const, $R_2\approx$ const. The eigenfunctions of stability problems of hinge-supported panels have the form

$$\varphi(x,\ y)=\varphi_0\sin k_m x\sin k_n y,$$
$$w(x,\ y)=w_0\sin k_m x\sin k_n y,\quad k_n=n\pi/a,\ k_m=m\pi/b,$$
$$n,\ m=1,\ 2,\ \ldots$$

The eigenfunctions for shells of revolution are also the following

$$\varphi(x,\ y)=\varphi_0\sin k_m x\cos k_n y,\qquad (1)$$
$$w(x,\ y)=w_0\sin k_m x\cos k_n y,$$

$k_n = n/R$, $k_m = m\pi/l$, n = 0, 1, ..., m = 1, 2, ... For n = 0, we obtain the eigenfunctions of axisymmetric buckling from the relationships (1). The eigennumbers of the problem under consideration are found from the formula

$$\lambda_{mn}=\frac{(k_m^2+k_n^2)^4+\varkappa^4\,(k_m^2+\chi k_n^2)^2}{(k_m^2+\vartheta k_n^2)\,(k_m^2+k_n^2)^2},\qquad (2)$$

where $\lambda=-\sigma\alpha_1/D$, $\varkappa^4=\mathrm{Eh}/DR_2^2$, $\chi=R_2/R_1$, $\vartheta=\alpha_2/\alpha_1$.

Let us introduce a polar coordinate system

$$k_m=r\cos\theta,\ k_n=r\sin\theta\ (r\geqslant 0,\ 0\leqslant 0\leqslant \pi/2)\qquad (3)$$

in the plane of the wave numbers k_m, k_n. After substituting these expressions for k_m, k_n into (2), we obtain a biquadratic equation in the polar radius r. After still another substitution $\xi=\sin^2\theta$ $(0\leqslant\xi\leqslant 1)$, the formula for r takes the form $(\eta=\lambda/2\varkappa^2)$

$$r_{1,2}^2=\varkappa^2\{\eta[1-\xi(1-\vartheta)]\pm\sqrt{\eta^2[1-\xi(1-\vartheta)]^2-[1-\xi(1-\chi)]^2}\}.\qquad (4)$$

The relationship (4) determines the boundary of the domain Ω within which $\eta<\eta_0$. It is meaningful under the condition

$$\eta^2[1-\xi(1-\vartheta)]^2-[1-\xi(1-\chi)]^2\geqslant 0.\qquad (5)$$

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 2, pp. 134-137, March-April, 1983. Original article submitted March 3, 1981.

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